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ON THE STABILITY OF STEADYSTATE MOTIONS

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Paper [1] describes a method for investigating the stability of steadystate motions of mechanical system. This method enables one to obtain the sufficient, and in some cases the necessary, stability conditions.

The present paper concerns certain aspects and further possibilities of the above method, including its applicability to nonholonomic systems. Its relationship to the Chetaev method for constructing Liapunov functions is considered. The discussion is illustrated with examples.

Let us consider some mechanical system whose phase variables characterizing its position and velocities at any instant t (or some of these variables) are x_s $(s = 1, \ldots, n)$. We assume that the variables x_s are independent if the system is holonomic, or that they may be related by some nonintegrable constraining equations if the system is nonholonomic. As these variables we can take, for example, the Lagrange variables of the system q_j, q_j; other possibilities are to take certain nonholonomic coordinates or quasi-coordinates.

Let us assume that some number of independent first integrals

$$F_i, (x_1, \ldots, x_n) = c_i \qquad (i = 1, \ldots, m, \quad m < n) \tag{1}$$

not explicitly dependent on time are known for the differential equations of motion of the system written in one way or another; c_i are arbitrary integration constants.

Let us recall the theorem of Routh [2] with Liapunov's important addendum [3].

Theorem. If some number of integrals not explicitly dependent on time has been obtained for the differential equations of motion of some system, and if among these integrals there is one which has a minimum or a maximum for all the given values of the remaining integrals as well as for all of their values which are sufficiently close to the given ones, and, finally, if the values of the variables in the integral which deliver its extremum are continuous functions of the values of these integrals, then the motion of the system for certain values of the variables which minimize or maximize the integral in question for the given values of the other integrals is stable with respect to these variables for all sufficiently small perturbations.

Liapunov did not prove this theorem, apparently regarding it as self-evident. It is possible, in fact, to adduce a very simple proof [4], whose idea can be stated briefly as follows.

Let $F_1(x_1, \ldots, x_n) = c_1$ be the integral referred to in the theorem.

Since, by hypothesis, this integral has a minimum or maximum both for given values of the constants $c_j = c_j^\circ$ and for all sufficiently close values $c_j = c_j^\circ + \Delta c_j$ $(j = 2, \ldots, m)$ of the remaining integrals, the function

$$V = F_1(x_1, \ldots, x_n) - F_1(x_1^*, \ldots, x_n^*) \tag{2}$$

is a sign definite function of the variables

$$\Delta x_s = x_s - x_s^*$$

which satisfies all the conditions of the Liapunov stability theorem $(V^{\cdot} \equiv 0)$. Here x_s^* denote the values of the variables x_s corresponding to the minimum or maximum of the function $F_1(x_1, \ldots, x_n)$ for the perturbed values of the constants $c_j = c_j^{\circ} + \Delta c_j^*$. By virtue of the continuous dependence of the values of x_s which minimize or maximize the function F_1 on the constants c_j of the remaining integrals, we can clearly choose our perturbations so small that the unperturbed motion $x_s = x_s^{\circ}$ of the system will lie in the neighborhood of the minimum or maximum of the function F_1 for the perturbed values of the constants $c_j = c_j^{\circ} + \Delta c_j^*$. This fact implies the stability of the unperturbed motion $x_s = x_s^{\circ}$ with respect to x_s for all sufficiently small perturbations.

Following [5], we shall call the Routh theorem together with Liapunov's addendum the 'Routh-Liapunov theorem'.

It is important to note that in proving this theorem we did not assume that the system was holonomic. This means that the Routh-Liapunov theorem is valid for both holonomic and nonholonomic systems.

In many problems in mechanics the integral which can have a minimum for the given values of the constants of the other first integrals is usually the integral $H = T - U =$ const of the system energy.

Another proof of the theorem with special reference to the energy integral for a holonomic system with cyclical coordinates is given in [6], where it is assumed that the Hamiltonian H is holomorphic in the neighborhood or the steadystate motion. A special case in which the Routh theorem is applicable to nonholonomic systems is considered in notice [7] using the Routh method for ignoring the cyclical coordinates.

If cyclical coordinates of the system are involved, ignoring them gives rise to an altered potential energy of the system, and the Routh-Liapunov theorem can be formulated somewhat differently, e.g. in the form of Theorem 4 of [4]. We must bear in mind, however, that there are cases when with a certain choice of the Lagrange coordinates q_j not all of the known first integrals (1) of the system linear with respect to the velocities $q_j^{\cdot}$ correspond to the cyclical coordinates. In such cases it is better to use the above formulation of the Routh-Liapunov theorem rather than the Routh theorem in the form of [8] or of Theorem 4 of [4] in dealing with the energy integral obtained by ignoring the cyclical coordinates.

We can illustrate this by means of an example.

Example 1. The equations of rotational motion of a symmetric projectile (cf. [8], pp. 108-111) admit of an energy integral $H =$ const and of two integrals $F_i =$ const $(i = 1,2)$ linear with respect to the velocities; only the second of these integrals, $p =$ const, corresponds to the cyclical coordinate ω. In the case of unperturbed motion $\alpha = \beta = \alpha^{\cdot} = \beta^{\cdot} = 0$ the function $W = a \cos \alpha \cos \beta$ does not have a minimum. However, the integral $H =$ const clearly has a minimum for fixed values of the two other integrals provided the Majewski condition $A^2p^2 - 4aB > 0$ is fulfilled.

In connection with this example we recall [9] that integrals linear with respect to the velocities $q_j^{\cdot}$ are exhibited only by holonomic systems which either have cyclical coordinates or can be transformed into systems with cyclical coordinates, which makes it possible to find an extended point transformation of the variables q_j into the variables Q_j such that all integrals of the form (1) linear with respect to the velocities correspond to the cyclical coordinates Q_α. In this case the formulations of the Routh-Liapunov theorem under consideration are clearly equivalent if the minimizable integral happens to be the energy integral.

Let us set down two addenda to the Routh-Liapunov theorem.

Addendum 1. The theorem remains valid even if motion of the system involves energy dissipation and if the energy integral does not exist, provided that the energy H of the system (or of part of the system) satisfies the remaining requirements of the theorem and that the derivative of H with respect to time is nonpositive by virtue of the equations of motion.

In fact, taking the energy H as our fonction F_1 and considering a function V of the form (2), we see that the above proof of the theorem remains valid, since $V^{\cdot} \leq 0$.

Example 2. The equations of motion along a horizontal plane of a heavy gyrostat with a spherical base (e.g. cf. [10]) admit of the energy relation $H \leq$ const (the equality sign applies in the case of an absolutely rough or an absolutely smooth plane) and of two integrals $F_i =$ const (4.10) and (4.11), of which the first corresponds to the cyclical coordinate ψ only if the center of mass of the gyrostat coincides with the center of the spherical base [7]. The function H has a minimum for fixed values of the first integrals $F_i =$ const if condition (4.18), i.e.

$$\left(C - A\frac{a}{l}\right) r_0^2 + C_2\omega r_0 + Mg\frac{a_1 l}{a} > 0 \qquad (l = a - a_1 > 0)$$

is fulfilled.

Addendum 2. The Routh-Liapuniv theorem remains valid if some of integrals (1) depend explicitly on time provided these integrals can be reduced by ordinary variable substitution to equations not containing time explicitly.

Example 3. The equations of motion of a dynamically symmetric gyrostat with a fixed point in the form of a solid body with a rotor acted on by external forces with the potential energy $V(\gamma_3)$ and by certain internal forces admit of the first integrals [10]

$$H = A\,(p^2 + q^2) + 2V(\gamma_3) = \text{const}, \qquad F_1 = A\,(p\gamma_1 + q\gamma_2) + [Cr + k\,(t)]\gamma_3 = \text{const},$$
$$F_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0, \quad F_3 = Cr + k\,(t) = \text{const}$$

The second and fourth of these integrals depend explicitly on time by way of the variable gyrostatic moment $k\,(t)$ which is determined by the internal forces and is assumed to be continuous. The function $V(\gamma_3)$ is assumed to be bounded and continuous and to have continuous and bounded first- and second-order derivatives.

Replacing the variable r by the new variable R according to the formula

$$Cr + k\,(t) = R$$

we reduce these integrals into integrals not explicitly dependent on time. Setting $R = R_0$ and considering the function $\Phi = H + 2\lambda F_1 + \mu F_2 \quad (\pm\mu = -\lambda R_0 V'(1))$ (λ is an arbitrary constant) we find that it has a time-independent value at the point determined by the following values of the variables:

$$p = q = 0, \ \gamma_1 = \gamma_2 = 0, \ \gamma_3 = \pm 1$$

This point corresponds to rotation of the gyrostat about the stationary axis ζ with the variable angular velocity

$$r = \frac{1}{C}\,(R_0 - k\,(t))$$

The minimum conditions for the function H reduce to the inequality

$$R_0^2 \mp 4AV'(1) > 0$$

which is a sufficient condition of stability of the gyrostat motion under consideration. Similar transient motions which depend explicitly on time are possible in the case of a symmetric gyrostatic satellite in a central Newtonian force field [4].

As is clear from the formulation of the Routh-Liapunov theorem, its application involves solution of the following three problems: 1) finding the time-independent points of the function F_1 for the given fixed values of the constants c_j ($j = 2, \ldots, m$) of the remaining first integrals (1) corresponding to the true motions of the system; 2) determination of the conditions under which a given time-independent point corresponds to the minimum or maximum of the function F_1; 3) verification of the requirement of continuous dependence on the quantities c_j of the coordinates of the points corresponding to the minimum or maximum of the function F_1.

We note that the conditions resulting from solution of the first two of these problems ensure (by virtue of the Routh theorem) the stability of the unperturbed motion at least for those perturbations which do not alter the quantities c_j ($j = 2, \ldots, m$). Fulfillment of the conditions of the third problem in addition to those of the first two ensures unconditional stability (by virtue of the Liapunov addendum).

By introducing the Lagrange multipliers λ_j we can reduce the first problem to that of the unconditional extremum of the function

$$\Phi(x_1, \ldots, x_n; \lambda_2, \ldots, \lambda_m) = F_1(x_1, \ldots, x_n) + \sum_{j=2}^{m} \lambda_j F_j(x_1, \ldots, x_n) \tag{3}$$

Let us assume that we have found values

$$x_s = x_s^\circ, \qquad \lambda_j = \lambda_j^\circ \tag{4}$$

satisfying Eqs.

$$\frac{\partial \Phi}{\partial x_s} = 0 \quad (s = 1, \ldots, n), \qquad F_j(x_1, \ldots, x_n) = c_j^\circ \quad (j = 1, \ldots, m) \tag{5}$$

We denote the value of the function Φ corresponding to values (4) by

$$\Phi^\circ = \Phi(x_1^\circ, \ldots, x_n^\circ, \lambda_2^\circ, \ldots, \lambda_m^\circ).$$

Let us consider the function

$$V = \Phi - \Phi^\circ = F_1 - F_1^\circ + \sum_{j=2}^{m} \lambda_j^\circ (F_j - F_j^\circ) \tag{6}$$

which, by virtue of Eqs. (5), clearly does not contain terms linear with respect to the variations of the variables x_s in the neighborhood of solution (4). If V under some conditions is a sign definite function in the neighborhood of the solution $x_s = x_s^\circ$, then the function Φ for the solution under consideration has an unconditional minimum, or maximum and the function f_1 has a conditional minimum or maximum. But the expressions

$$V_j = F_j - F_j^\circ = \text{const}$$

are clearly the first integrals of the equations of perturbed motion of the system, as is function (6), which can be rewritten as

$$V = V_1 + \sum_{j=2}^{m} \lambda_j^\circ V_j \tag{7}$$

The form of this function points to a relationship between the Routh-Liapunov theorem and the effective method of Chetaev for constructing the Liapunov functions as a bundle of known first integrals $V_j = \text{const}$ of the equations of perturbed motion. The Chetaev method consists in constructing a Liapunov function of the form

$$V = V_1 + \sum_{j=2}^{m} \lambda_j V_j + \sum_{\alpha=1}^{m} \mu_\alpha V_\alpha^2 \qquad (8)$$

where the constants λ_j $(j = 2, \ldots, m)$, μ_α $(\alpha = 1, \ldots, m)$ are chosen in such a way that (8) is a sign definite function. By the choice of the multipliers λ_j, neither (8) nor (6) contain terms linear with respect to the variations of the variables x_s (otherwise they could not be the sign definite expressions). It is clear that with the coefficients λ_j chosen in this way, their role in Expression (8) is similar to that of the Lagrange multipliers in variational problem (6).

In the case where all of the $\mu_\alpha (\alpha = 1, \ldots, m)$ are equal to zero and where $\lambda_j = \lambda_j^\circ$ $(j = 2, \ldots, m)$, Liapunov function (8) is identical with function (7). If (7) also happens to be a sign definite function then (since $V^\cdot \equiv 0$) it satisfies all the conditions of the Liapunov stability theorem. Hence, the sign definite conditions for function (7) which, as noted above, ensure the conditional minimum or maximum of the function F_1, are sufficient conditions of unconditional stability of unperturbed motion. Because of this, the third of the above problems (i.e. the verification of continuous dependence of the coordinates of the extremum points of the function F_1 on the values of c_j) is in this case superfluous.

We note that the sufficient conditions of a conditional minium or maximum of the function F_1 are also obtainable from the sign definite conditions for function (7) on linear manifolds [11]

$$\sum_{s=1}^{n} \frac{\partial F_j}{\partial x_s} (x_s - x_s^\circ) = 0 \qquad (j = 2, \ldots, m) \qquad (9)$$

The number of conditions (9) is smaller than the number of conditions for arbitrary values of the variations of x_s. The difference between these numbers amounts to $m - 1$ conditional independent Eqs. (9). In general, the latter can be broader than the conditions of an unconditional extremum of this function [5].

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